※ Information 3: Hidden Variable Theories and the CHSH Game

"[Quantum mechanics] delivers much but it hardly brings us closer to the Old One's secret. In any event, I am convinced that He is not playing dice."

- Albert Einstein

Classical physics allow us to predict the exact trajectories and outcomes of physical systems, given that we know enough information in the beginning. We model coin flips as being events that have a 0.5 probability of landing in heads or tails, but in principle, if we had enough precise control over our coinflip setup, we could predict exactly where the coin would land.

Quantum physics however, does not seem to have this property. As we discussed before, if Alice and Bob share a qubit from a Bell pair, no matter how far away they are, Alice's measurement result seems to affect what Bob's measurement result is. Stranger still is that the initial measurement (from either of them) is completely random! Maybe this is just a limitation of our knowledge of quantum mechanics, and in principle it is possible for us to perfectly predict which state the pair will collapse to given perfect information...

Question 76. What happens to Alice's qubit if Bob measures his qubit from the Bell pair in some arbitrary basis $\{|v\rangle, |w\rangle\}$ and sees $\{|w\rangle\}$?

Question 77. Why is it problematic that Alice's measurement result affects Bob measurement result no matter how far they are separated?

To reconcile this conflict, scientists hypothesized that particles had "hidden variables" representing the possible outcomes they would be in if they were measured. One computer science flavored hidden variable theory might hypothesize the following: When Alice and Bob's qubits first get entangled, they decide what their measurement result will be across all possible bases. If they are too far apart to communicate fast enough, they will just consult the local list to decide what state to collapse to.

Question 78. As a computer scientist, how would you argue that the above hidden variable theory is not reasonable?

John Bell and others designed an experiment which ruled out most known hidden variable theories, including the one we mentioned above. Here, ruling out means that the experiment behaves in a way which would contradict the assumption that a hidden variable theory exists. Instead of looking at the experiment, we will study a game that isolates the critical parts.

9.1 The CHSH Game

Here are the rules of the game:

- Alice and Bob will be separated with no way to communicate with each other.
- Charlie will flip two coins, and send a bit to Alice and Bob depending on the outcomes. Let's say he sends a 0 if tails, and a 1 if heads.
- After receiving the random bit x, Alice sends back an answer bit a to Charlie.
- After receiving the random bit y, Bob sends back an answer bit b to Charlie.
- Alice and Bob win against Charlie if the answer bits and random bits satisfy

$$a + b \pmod 2 = xy. \tag{61}$$

Question 79. If Charlie flipped heads for Alice's bit, and tails for Bob's bit, what is a choice for Alice and Bob's response for them to win?

Question 80. If Charlie flipped heads for both coins, what is a choice for Alice and Bob's response for them to win?

x	y	xy	Win condition
0	0	0	
0	1	0	
1	0	0	
1	1	1	

Alice and Bob can discuss a strategy and share as many bits of information as they would like with each other, but they are not allowed to communicate once the game begins.

Question 81. If Alice and Bob decide to use the strategy that they *always* output $a = 0$ and $b = 0$, what is the probability that they win?

We won't prove it here, but the above winrate is the best classical strategy for Alice and Bob. A hidden variable theory is trying to explain quantum mechanics in a classical way, so if qubits do store hidden variables, there should be no way to consistently beat the CHSH game with a probability higher than we calculated above.

For the remainder of the course, we will use the following shorthand to describe a state with real amplitudes:

$$|\theta\rangle := \cos\theta\,|0\rangle + \sin\theta\,|1\rangle \tag{62}$$

Question 82. Draw the two bases $\{|\pi/6\rangle, |2\pi/3\rangle\}$ and $\{|-\pi/6\rangle, |\pi/3\rangle\}$ in the $|0\rangle$, $|1\rangle$ plane.

Question 83. Suppose the state $|+\rangle$ is measured in the $\{|\pi/6\rangle, |2\pi/3\rangle\}$ basis. What is the probability of each outcome? Express your answer as a function of $\cos\theta$ and $\sin\theta$.