※ Foundations 2c: Two Qubits, Tensor Products, and Entanglement

Entanglement is not one but rather the characteristic trait of quantum mechanics.

- Erwin Schrodinger

The final building block for quantum systems that we need before going forward is **entanglement**. In simple terms, entanglement captures how dependent the probability of seeing one state is to the probability of seeing another state. It is a quantum generalization of joint and conditional distributions.

6.1 Partial Measurements

Once we have multiple qubits, we can decide to just measure one qubit out of the full system. Suppose Alice and Bob each have one qubit, and an operation is performed on the full system (both qubits). Is there something we can say about Bob's qubit by **only** measuring Alice's qubit?

We will use the following state to illustrate examples in this section:

$$|\psi\rangle := \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \alpha |00\rangle + \beta |01\rangle + \gamma |10\rangle + \delta |11\rangle . \tag{45}$$

We will say the first qubit is Alice's.

Question 47. If Alice performs a measurement in the standard basis, what is the probability that she observes $|0\rangle$? What is the state after the measurement? Can the state be written as a tensor product of two states?

Definition 6.1 (Partial Measurement Rule)**.** A quantum system collapses to fit the observed outcome from the measurement made.

Sum the squares of coefficients that agree w/ that outcome.

Prob. Alice sees $|0\rangle \rightarrow |\alpha|^2 + |\beta|^2$

State after $\rightarrow \dfrac{\alpha|00\rangle + \beta|01\rangle}{\sqrt{|\alpha|^2+|\beta|^2}}$

Divide by sqrt of Length.

→ **Question 48.** Write the state $|+-\rangle := |+\rangle \otimes |-\rangle$ in the standard basis. What's the probability that Alice sees a $|0\rangle$ after measuring in the standard basis?

State in S.B.

$$\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) = \underbrace{\frac{1}{2}}_{\alpha}|00\rangle - \underbrace{\frac{1}{2}}_{\beta}|01\rangle + \underbrace{\frac{1}{2}}_{\gamma}|10\rangle - \underbrace{\frac{1}{2}}_{\delta}|11\rangle$$

Prob. of seeing $|0\rangle$

$$|\alpha|^2 + |\beta|^2 = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

State after

$$\left(\frac{1}{2}|00\rangle - \frac{1}{2}|01\rangle\right) \Big/ \sqrt{1/2} = \frac{1}{\sqrt{2}}|00\rangle - \frac{1}{\sqrt{2}}|01\rangle$$

$$= |0\rangle\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) = |0\rangle|-\rangle$$

Question 49. Consider the following two states.

- $|\phi_1\rangle := \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$
- $|\phi_2\rangle := \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$

Express $|\phi_1\rangle \otimes |\phi_2\rangle$ in the standard basis. What's the probability that Bob see's $|1\rangle$ after measuring in the standard basis?

State in S.B.

$$|\phi_1\rangle \otimes |\phi_2\rangle = \left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right) \otimes \left(\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle\right)$$

$$= \underbrace{\frac{1}{4}}_{\alpha}|00\rangle \boxed{\underbrace{-\frac{\sqrt{3}}{4}}_{\beta}|01\rangle} + \underbrace{\frac{\sqrt{3}}{4}}_{\gamma}|10\rangle \boxed{\underbrace{-\frac{3}{4}}_{\delta}|11\rangle}$$

Prob. of Bob seeing $|1\rangle$

$$|\beta|^2 + |\delta|^2 = \frac{3}{16} + \frac{9}{16} = \frac{12}{16} = \frac{3}{4}$$

State after if he does.

$$\left(-\frac{\sqrt{3}}{4}|01\rangle - \frac{3}{4}|11\rangle\right) \cdot \frac{\sqrt{4}}{\sqrt{3}} = -\frac{1}{\sqrt{4}}|01\rangle - \frac{\sqrt{3}}{\sqrt{4}}|11\rangle$$

$$= -\frac{1}{2}|01\rangle - \frac{\sqrt{3}}{2}|11\rangle$$

$$= \left(-\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle\right) \otimes |1\rangle$$

$\boxed{\frac{1}{\sqrt{2}}|00\rangle + 0\,|01\rangle} + 0\,|10\rangle + \frac{1}{\sqrt{2}}|11\rangle$

If a 2 qubit state can be written as a tensor product, we call the state **independent** or **separable**. Not every two qubit state is separable, with an important example being the following state.

$$\text{Bell pair} \rightarrow \frac{|00\rangle + |11\rangle}{\sqrt{2}}. = \begin{array}{cl} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix} & \begin{array}{l} \leftarrow \alpha \\ \leftarrow \beta \\ \leftarrow \gamma \\ \leftarrow \delta \end{array} \end{array} \tag{46}$$

This state is so important in quantum information, that you'll see it referred to by different names including the **singlet state**, **Bell pair**, and others. When a state cannot be written as a tensor product of single qubit states, we say that the state is **entangled.**

Question 50. If we measure the first qubit in a Bell pair, what is the probability of observing $|0\rangle$? What is the state of our system afterwards?

$\frac{1}{2}$, state after $|00\rangle$

Question 51. Let $|\psi\rangle := \left(\frac{1}{\sqrt{3}}|0\rangle - \sqrt{\frac{2}{3}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$. Suppose we measure the first qubit and observed $|0\rangle$. What was the probability of this event occurring? What is the state of the system after?

Question 52. Let $|\phi\rangle := \frac{1}{3}|00\rangle + \frac{1}{\sqrt{3}}|10\rangle + \frac{1}{\sqrt{3}}|11\rangle$. Suppose we measure the first qubit and observed $|0\rangle$. What was the probability of this event occurring? What is the state of the system after? What if we observed $|1\rangle$?

6.2 Two Qubit Operations

We will now define operations on two qubit states. Since the space is now 4 dimensions, these actions are represented by 4×4 unitaries.

Definition 6.2 (CNOT gate)**.** The CNOT (Controlled-NOT) gate will flip the second qubit if the first qubit is $|1\rangle$, and otherwise do nothing.

$|00\rangle \xrightarrow{CNOT} |00\rangle$

$|01\rangle \xrightarrow{CNOT} |01\rangle$

$|10\rangle \xrightarrow{CNOT} |11\rangle$

$|11\rangle \xrightarrow{CNOT} |10\rangle$

$$CNOT := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{47}$$

What if we want to apply a gate to only one qubit? The tensor product formalism gives us a nice way to define this. Suppose we want to apply an X gate to just the second qubit. We can represent this action as applying an I gate to the first qubit.

$$I \otimes X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 0 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ 0 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 1 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{48}$$

In general, if we have a separable state $|\psi_1\rangle \otimes |\psi_2\rangle$ and two independent operators A and B, we can represent the action $A \otimes B$ as follows:

$$(A \otimes B) |\psi_1\rangle \otimes |\psi_2\rangle = A |\psi_1\rangle \otimes B |\psi_2\rangle. \tag{49}$$

Question 53. Use the above shortcut to compute the action of $H \otimes H$ on all four standard basis states.

$(H \otimes H)(|0\rangle \otimes |0\rangle) = (H|0\rangle) \otimes (H|0\rangle) = |+\rangle |+\rangle$

Sometimes it will be easier to work in bra-ket notation rather than using matrices, especially as the size of the system increases. Consider the operation $I \otimes I \otimes H \otimes I$ on a system of 4 qubits. The matrix needed to represent this has a size of 16×16. Instead, we can simply analyze the effect it has on states that are relevant:

$$I_1 \otimes I_2 \otimes H_3 \otimes I_4 \, |0000\rangle = |00+0\rangle \tag{50}$$

$$= |00\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle \tag{51}$$

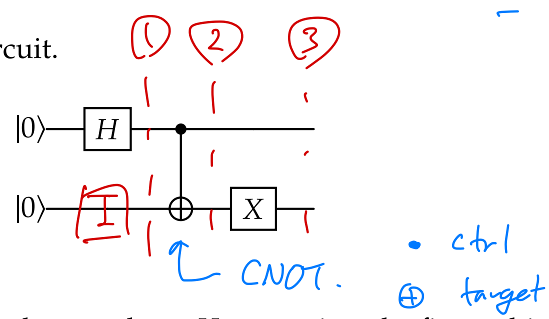
$$= \frac{1}{\sqrt{2}}(|0000\rangle + |0010\rangle). \tag{52}$$

By linearity of operators, we can just perform the analysis on each term independently before combining them together at the end.

6.3 Qubits in Quantum Circuit Notation

$- H \cdot H -$

Consider the following two qubit circuit.



The circuit starts in the state $|00\rangle$. We then apply an H gate to just the first qubit, and then a CNOT gate with the first qubit as the control (a black circle represents the control bit).

Question 54. What is the state of the system after each operation?

1 $(H \otimes I)\,|00\rangle = (H|0\rangle) \otimes (I\,|0\rangle) = |+\rangle \otimes |0\rangle = \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |0\rangle$

2 $\mathrm{CNOT}\left(\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) \otimes |0\rangle\right) = \mathrm{CNOT}\left(\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|10\rangle\right)$

$= \frac{1}{\sqrt{2}}(\mathrm{CNOT}\,|00\rangle) + \frac{1}{\sqrt{2}}(\mathrm{CNOT}\,|10\rangle)$

$= \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle.$ → Bell pair!

3 $\frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle$

6.4 Spooky Action at a Distance

Einstein never fully accepted quantum mechanics, in part because of the mysterious effects of entanglement. Once a Bell pair is created, in theory the corresponding qubits can be physically separated infinitely far. Experimentally, this has been verified up to 1000 miles apart!

Why did this fact not sit well with Einstein and other physicists of the time? Suppose we created a Bell pair and moved one qubit to a different galaxy. If Alice now measures her qubit, Bob's state instantly collapses to the same state! This happens faster than the speed of light and Alice has no way of communicating her result to Bob. How does Bob's state know what to collapse to? Is it somehow "predetermined" by the two states?

Question 55. Suppose Alice and Bob decide to measure in the Hadamard basis instead. What is the probability that Alice observes $|+\rangle$, and what is Bob's state afterwards?

Scientists referred to this and similar effects as "hidden variable theory", claiming that somehow the states agree upon how to be measured. We will later prove that there is no hidden variables, and the entanglement is really happening in the way we modeled!