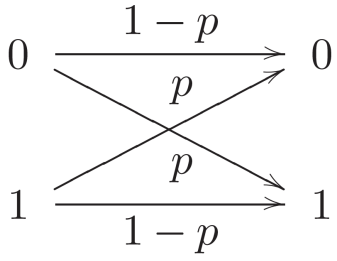
※ Information 4: Error Correction

We've learned a bit more about the theory behind quantum information, and we know (by the existence of this course) that we will be learning about some algorithms in which quantum computers provide some advantage. Many people are working on building quantum computers that work in order to do all the things we talk about in this course, but why is it taking so long?

Computing (even classically) is susceptible to **errors**. Imagine a bit is being stored on a hard drive in your laptop. The bit starts out in the state 0 or 1, but after a long time it becomes likely that stray magnetic fields causes the bit to flip. Or, imagine we are sending a bit over a channel, but we know that there is a chance that messages get corrupted due to outside interference. We can model the effect of the noise in the channel using a probability $p > 0$ of the bit flipping.



To protect information against the effects of noise, we **encode** the information by adding some extra information to the message. That way, even if errors were to occur, we hope that we have enough information remaining to **decode** the message.

Question 86. Consider the **repetition code** where we encode the bits 0 and 1 as follows.

$$0 \to 000 \tag{64}$$

$$1 \to 111 \tag{65}$$

If the probability of error on one bit is $p = 0.25$, what is the probability that we are able to reconstruct the correct message?

Quantum states are also vulnerable to errors. There are a few challenges that are unique to quantum states compared to classical ones.

1. **No cloning**: We can't do a complete replication code like we did for classical due to the No-cloning theorem.
2. **Errors are continuous**: We can think of a bit-flip error as an application of an X gate. We can model errors using other gates too, which means that in principle there are infinitely many different types of errors that could occur.
3. **Measurement destroys quantum information**: Error correction requires knowing whether an error occurred or not. How would we know that if we don't make a measurement?

10.1 Projective Measurements

Suppose we have a qubit in the state $|\psi\rangle$. If we want to find the probability of seeing $|0\rangle$ when measuring in the standard basis, we need to compute $|\langle\psi|0\rangle|^2$. We would like to rewrite this in a particular way...

$$\text{Probability of seeing } |0\rangle = |\langle\psi|0\rangle|^2 \tag{66}$$

$$= \langle\psi|0\rangle\langle 0|\psi\rangle \tag{67}$$

We call the matrix in the middle a **projector** on the $|0\rangle$ state. We can think of it as the matrix that tells us how similar $|\psi\rangle$ is to $|0\rangle$, but ultimately is just one way to rewrite the way we have been doing measurements up to this point. Note that this is a method of finding the probabilities, and a measurement still requires an orthogonal set of projectors.

Question 87. What is the projector on the $|1\rangle$ state? How about the projector on the $|+\rangle$ state? What's the state after seeing $|1\rangle$?

10.2 The three qubit bit flip code

Let's create a three qubit code that just needs to detect whether a bit flip has occurred or not. We will assume that the error occurs with probability p.

Question 88. Suppose we have a qubit in the state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$. We will encode this qubit as $\alpha|000\rangle + \beta|111\rangle$. Note that this does not require cloning the state, so it is allowed! Draw a three qubit circuit that starts in the state $|\psi\rangle|0\rangle|0\rangle$ and ends in $\alpha|000\rangle + \beta|111\rangle$.

For this code, we will call $|000\rangle = |0_L\rangle$ the "logical $|0\rangle$" and $|111\rangle = |1_L\rangle$ the "logical $|1\rangle$" as opposed to the "physical" $|0\rangle$ and $|1\rangle$.

Question 89. We will have a total of four projection operators representing the different "types" of bit-flip errors that could occur (that we can distinguish). Label each of the following projection operators with what they represent.

$$P_0 \equiv |000\rangle\langle 000| + |111\rangle\langle 111| \tag{68}$$

$$P_1 \equiv |100\rangle\langle 100| + |011\rangle\langle 011| \tag{69}$$

$$P_2 \equiv |010\rangle\langle 010| + |101\rangle\langle 101| \tag{70}$$

$$P_3 \equiv |001\rangle\langle 001| + |110\rangle\langle 110| \tag{71}$$

Question 90. Suppose that a bit flip occurs on qubit one, so we now have the corrupted state

$$|\psi\rangle = \alpha\,|011\rangle + \beta\,|100\rangle\,. \tag{72}$$

Calculate the result of the projective measurements for P_0 and P_1 on this state, and verify that these projective measurements can tell us information about an error occurring.

Question 91. Use the following identities to determine what the state $|\psi\rangle$ from the previous problem is after a projective measurement using P_1:

$$p_1 = \langle\psi|P_1|\psi\rangle\,, \tag{73}$$

$$\text{State after measurement: } \frac{P_1\,|\psi\rangle}{\sqrt{p_1}} \tag{74}$$

10.3 Observables

We will take a quick detour to introduce the "observable" picture. An observable describes some physical property that we are able to measure, and we model this mathematically using a **Hermitian matrix** A. Such a matrix is defined as one that has **real-valued eigenvalues**. In general, we will write a projective matrix as

$$A = \sum_m a_m P_m \tag{75}$$

where

- a_m is the set of eigenvalues of A, and
- P_m is the set of projectors onto the "eigenspaces" of A.

Here, we should think of a_m as the distinguishable outcomes of the measurement, that tell us which P_m we are in.

Let's do a single qubit example. It turns out that the Z operator represents an observable.

Question 92. What are the eigenvalues and eigenvectors of Z? Remember, an eigenvalue/eigenvector pair of a matrix is a pair of number a and vector $|p\rangle$ such that $Z|p\rangle = a|p\rangle$.

Now let's consider the observable $Z \otimes Z \otimes I$. I'll use the shorthand $Z_1 Z_2$ denoting which qubits the Z observable is being measured on.

Question 93. What are the eigenvalues and eigenvectors of the observable $Z_1 Z_2$?

Question 94. Suppose a bit flip error occurred on our repetition code, creating the state $\alpha |010\rangle + \beta |101\rangle$. What are the measurement outcomes if we measure the observable $Z_1 Z_2$, and then $Z_2 Z_3$? What is the state after the measurement?

10.4 The three qubit phase flip code

Bit flips are just one type of error that can occur for a quantum state! For example, we might be working with a system where a *phase flip* error, occurs with probability p. More precisely, with probability $p > 0$, the phase flip operator Z is applied to our state, taking a state $a|0\rangle + b|1\rangle$ to the state $a|0\rangle - b|1\rangle$.

Question 95. Why is it problematic to encode our logical qubits in the same way as we did for the bit flip code? That is, representing the logical states as $|0_L\rangle = |000\rangle$ and $|1_L\rangle = |111\rangle$. Consider the action of the phase flip error on the state $a|0_L\rangle + b|1_L\rangle$.

Question 96. Suppose we have a quantum system where we know that the only type of error that can occur is a phase flip error. How should we encode the logical qubit $|0_L\rangle$ and $|1_L\rangle$? Draw the circuit that prepares this state.