CS166 WI24: Homework 1 (Due Jan 16 11:59pm)

※ Problem 1

Frankie the frog lives in a pond with two lily pads, east and west. One day, she found two coins at the bottom of the pond and has placed one on each of the two lily pads. Every morning, Frankie flips the coin on the lily pad she spent her last day on, and jumps to the other lily pad if it lands heads. If the coin is tails, she stays on her lily pad for the day.

The state space is E and W, corresponding to the lily pad Frankie spends her day on. We cannot assume that the coins are fair, as we do not know where they came from! They could be weighted in very different ways. Let's call the probability that the east coin lands on heads to be p and the probability that the west coin lands on heads to be q.

1. On day one, Frankie is on the east lily pad. How can we express this fact using a probability vector?
2. Write down the stochastic matrix that corresponds to Frankie's game.
3. Gently place Frankie on the east lily pad and leave the pond for 2 days. Write down the probability vector that represents our best guess for Frankie's location.
4. If we left Frankie for an infinite number of days (of course Frankie lives forever!), what would our estimate of her location (the probability vector) converge to?
5. (Challenge) Come up with a strategy for estimating what p and q are. How many days will you need to have a confident estimate?
6. (Bonus) Draw Frankie the frog.

※ Problem 2

My friend had an incredibly successful Valorant season, achieving a 56.2% win rate over 200 games. However, she claims that she is not a good player, but an average one, and that her winrate was just her getting lucky.

We will define an **average player** as someone who has an expected win rate of exactly 50%.

What is the probability that she was an average player, using Markov's inequality? What if we used Chebyshev's inequality? Chernoff bound?

(Bonus) Write some code to experimentally see how likely or unlikely this event is.

※ Problem 3

1. Draw the two square roots of 1 on the complex plane.

2. Draw the three cubic roots of 1 on the complex plane, and express them in both the standard and phase representation.

3. Let $1, \omega_3, \omega_3^2$ be the three cubic roots you found in the previous problem. We call these the **3rd roots of unity**. The notation suggests that the third root of unity is the square of the second one. Try to use the geometry of the complex plane and the variables in the phase representation to show that this is true.

4. Pick some prime number p, and let ω_p be the first p-th root of 1 that isn't equal to 1, going clockwise from 1 on the complex plane. The **p-th roots of unity** is the set of numbers $\{1, \omega_p, \omega_p^2, \ldots, \omega_p^{p-1}\}$. Show that for any $k < p$, there exist integers s and t such that

$$(\omega_p^k)^s = \omega_p^t. \tag{1}$$

   That is, every p-th root of unity can be raised to a power to become a different p-th root of unity.

   Is this true if p is not prime? If it is, prove it, otherwise, come up with a counterexample.

   See if you can show the above statements geometrically on the complex plane.

※ Problem 4

We will consider the following quantum state

$$|\psi\rangle := \left(\frac{\sqrt{3}}{4} + i\frac{1}{4}\right)|0\rangle + \left(-\frac{\sqrt{3}}{4} + i\frac{3}{4}\right)|1\rangle. \tag{2}$$

1. Verify that $|\psi\rangle$ is a quantum state.

2. What is the probability that we measure $|0\rangle$ if we measure in the standard basis? What is the state after the measurement?

3. Find a state orthogonal to $|\psi\rangle$.

4. Consider the state $e^{i\pi/4}|\psi\rangle$. That is, multiply the two amplitudes by $e^{i\pi/4}$. What is the probability that we measure $|0\rangle$ if we measure in the standard basis?

※ Problem 5

Solve Question 8 from the lecture notes. It is restated here for convenience:

Suppose both of the coins begin in the state tails. Is there a sequence of actions on the individual coins (think stochastic matrices) such that the final state of the two coins will be B? Here,

$$B := \begin{bmatrix} 1/2 \\ 0 \\ 0 \\ 1/2 \end{bmatrix}. \tag{3}$$

※ Bonus Reading

In each homework, I will try to find a few external reading sources for you to explore if you want to learn more. If there were other resources which you found were helpful or interesting, please share them on Ed as well!

- Wikipedia article for the Chernoff-Hoeffding bound
- Alistair Sinclair Chernoff notes: A proof of a Chernoff bound. The proof highlights the intuition behind each step of the math, but some steps will be beyond the scope of what we cover in this class. Nontheless, it is a good resource to look at!
- Michael Goemans Chernoff notes: An alternate proof of a different version of the Chernoff bound. This one uses moment generating functions, and also connects it to Markov and Chebyshev inequalities.
- Simplified Chernoff bounds with powers-of-two probabilities: A new paper by two professors at UCI, Michael Dillencourt and Michael Goodrich. They introduce a simplified Chernoff bound with a proof for it that is more applicable to CS people.

※ Open Questions

I will also try to come up with some questions that are more research oriented. My goal is for them to be interesting, related to that week's content, and accessible for an undergraduate student. Note that this is a research level problem, so it is impossible to know how hard it really would be until it is attempted. Feel free to tackle them with classmates or discuss with me!

- **Is there a simple Chernoff bound that is easy to prove?** I initially wanted to assign proving a Chernoff bound as a homework problem. I think that a Chernoff bound that only applies to Bernoulli trials could be really useful and also have a simple proof. When I say easy, I mean that the average CS student should be able to solve a guided homework problem related to it after one week of this class.