CS166 WI24: Homework 2 (Due Wednesday Jan 24 11:59pm)

※ Problem 1

Let $|\psi\rangle = \frac{1}{2}|0\rangle + \frac{1-\sqrt{2}i}{2}|1\rangle$.

In class, we've seen the standard basis $\{|0\rangle, |1\rangle\}$ and the Hadamard basis $\{|+\rangle, |-\rangle\}$ for single qubit states. Let's define a third basis $\{|i\rangle, |-i\rangle\}$ where

$$|i\rangle := \frac{|0\rangle + i|1\rangle}{\sqrt{2}} \tag{1}$$

$$|-i\rangle := \frac{|0\rangle - i|1\rangle}{\sqrt{2}}. \tag{2}$$

1. What is the probability of seeing $|0\rangle$ and $|1\rangle$ if we measure $|\psi\rangle$ in the standard basis?
2. What is the probability of seeing $|+\rangle$ and $|-\rangle$ if we measure $|\psi\rangle$ in the Hadamard basis?
3. What is the probability of seeing $|i\rangle$ and $|-i\rangle$ if we measure $|\psi\rangle$ in the $\{|i\rangle, |-i\rangle\}$ basis?

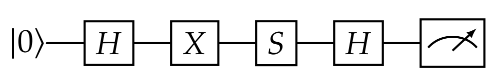
※ Problem 2

Consider the following quantum circuits and write down what the measurement probabilities are for each possible outcome. I've listed the standard gates that are often used in the literature here for convenience.

$$X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{3}$$

$$H := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad R_{\pi/4} := \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad S := \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \quad T := \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \tag{4}$$

1. Measure in the standard basis.



2. Measure in the Hadamard basis.

$$|+\rangle - \boxed{Z} - \boxed{R_{\pi/4}} - \boxed{Y} - \boxed{R_{\pi/4}} - \boxed{\text{measure}}$$

3. Measure in the standard basis.

$$|+\rangle - \boxed{T} - \boxed{H} - \boxed{\text{measure}}$$

4. Measure in the $\{|i\rangle, |-i\rangle\}$ basis.

$$|-\rangle - \boxed{Y} - \boxed{S} - \boxed{R_{pi/4}} - \boxed{Y} - \boxed{\text{measure}}$$

5. Find a circuit that inverts the action of the circuit in the first part of this problem. You may need to define you own gates to do this.

※ Problem 3

Solve question 20 from the homework. It is restated here for convenience.

Suppose we have an experiment whose outcome is either $|0\rangle$ (success) or $|+\rangle$ (failure), each occuring with probability $1/2$. In class, we saw that measuring in the standard basis and the Hadamard basis would not be able to distinguish the two states $1/4$ of the time. Is there a basis we can measure in such that the probability of failing is smaller than $1/4$?

※ Problem 4

Suppose we repeat the Elitzur-Vaidman bomb experiment but adjust our circuit as follows:

$$|0\rangle - \boxed{R_{\pi/6}} - \boxed{B} - \boxed{R_{\pi/6}} - \boxed{B} - \boxed{R_{\pi/6}} - \boxed{\text{measure}}$$

The final measurement is in the standard basis. Recall that

$$R_\theta := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{5}$$

To recap, the action of the "bomb" gate is the following:

- If the bomb is faulty, it does nothing and acts as an identity gate.
- If the bomb is not faulty, and the input state is $\alpha |0\rangle + \beta |1\rangle$, with probability $|\beta|^2$ the bomb explodes and the experiment ends. With probability $|\alpha|^2$, nothing happens and the state is reset to $|0\rangle$.

  Just like in class, there are three possible outcomes.

1. The bomb explodes
2. We measure $|0\rangle$
3. We measure $|1\rangle$

1. What are the possible outcomes and probabilities of those outcomes if the bomb is faulty?
2. What are the possible outcomes and probabilities of those outcomes if the bomb is not faulty?
3. How does this version of the experiment compare to what we did in class? Which is more effective at detecting a bomb that is not faulty without having it explode? Which has a higher probability that the bomb explodes?

❊ Problem 5

Consider the following 3 qubit state.

$$|\phi\rangle = \frac{\sqrt{3}}{4}|000\rangle + \frac{1}{4}|001\rangle + \frac{1}{2\sqrt{2}}|010\rangle + \frac{1}{2\sqrt{2}}|011\rangle + \frac{1}{4}|100\rangle + \frac{\sqrt{3}}{4}|101\rangle + \frac{1}{2\sqrt{2}}|110\rangle + \frac{1}{2\sqrt{2}}|111\rangle \tag{6}$$

1. Suppose we measure all three qubits. What is the probability that we see $|000\rangle$? What is the state of the system after the measurement?
2. Suppose instead we only measure the first qubit, and get the result $|1\rangle$. What is the probability of this occurring? What is the state after the measurement?

※ Problem 6

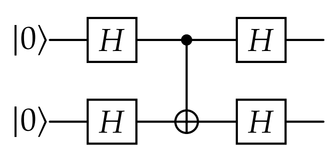
Give the result of applying the following two qubit gates on the following state:

$$|\psi\rangle := \frac{1}{\sqrt{2}}|01\rangle + \frac{1}{\sqrt{2}}|10\rangle \,. \tag{7}$$

1. $X \otimes I$
2. $I \otimes Z$
3. $X \otimes Z$

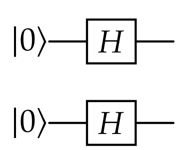
※ Problem 7

1. What does the CNOT gate do if the control qubit is in the state $|+\rangle$?
2. What does the CNOT gate do if the control qubit is in the state $|-\rangle$?
3. Show that the following circuit is equivalent to a single CNOT gate, but with the control qubit swapped with the CNOT gate seen here.

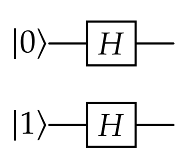


※ Problem 8

1. What is the result of the following circuit?



2. What is the result of the following circuit?



3. What is the result of the following circuit?

$$\begin{array}{l} |0\rangle - \boxed{H} - \\ |0\rangle - \boxed{H} - \\ |0\rangle - \boxed{H} - \\ |0\rangle - \boxed{H} - \end{array}$$

4. If we had n qubits and applied a Hadamard gate to each of them, what is the resulting state?

❊ Bonus Reading

- Wikipedia article for the Bloch sphere: Some of you may be wondering how to visualize a quantum state, and the effect of a quantum gate on a given state. There is a standard visualization for single qubit states called the Bloch sphere. It is quite effective for this setting, but I left it out of this course because it does not generalize well to multiple qubit states.

❊ Open Questions

- **Is there a natural way to visualize quantum states?** Geometric intuition is a powerful tool we have as humans, but the space of quantum states is shrouded in mystery as we do not know how to navigate the space well. Is there a way to generalize the Bloch sphere to visualize multiple qubit states better? The following paper seems to give a starting point using two qubits. https://arxiv.org/abs/2003.01699