CS166 WI24: Homework 3 (Due Tuesday Feb 6 11:59pm)

Problems 1 and 2 are from Ryan O'Donnell's 2018 offering of a quantum computing course.

※ Problem 1: The Perfect Magic Trick

Alice is on Mars, Bob is on Jupiter, Charlie is on Saturn. With each of them is a referee. The referees have agreed to pick one of the following four strings: 111, 100, 010, 001. The first bit will be presented to Alice, the second to Bob, and the third to Charlie.

Upon receiving their bit, Alice, Bob, and Charlie must quickly respond with a 0 or a 1. They "succeed with the magic trick" if:

The referees presented 111:The responses have an even number of 1's

The referees presented a single 1 and two 0's:The responses have an odd number of 1's

We assume that the spatial distance between the three parties prevent them from communicating at all.

1. Design a deterministic (classical) strategy for Alice, Bob, and Charlie in which the probability that they succeed is 3/4.
2. Suppose that Alice, Bob, and Charlie prepare the following 3-qubit state on Earth before the magic trick begins:

$$\frac{1}{2}|000\rangle - \frac{1}{2}|011\rangle - \frac{1}{2}|101\rangle - \frac{1}{2}|110\rangle . \tag{1}$$

   Alice takes the first qubit, Bob the second, and Charlie the third qubit to their respective planets. Now, when they receive their challenges, they each use the following strategy:

   - If they are presented with a 1, they measure their qubit and respond with the outcome.
   - If they are challenged with a 0, they first apply a Hadamard gate to their qubit, and then measure and respond with the outcome.

   What is the probability that Alice, Bob, and Charlie succeed with the magic trick?

※ Problem 2: Hardy's Paradox

Alice and Bob prepare the following two qubit state:

$$|\psi\rangle = (H \otimes H)\left(\frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|10\rangle\right). \tag{2}$$

Note that we are applying a Hadamard gate to each of the two qubits. Alice takes the first qubit and Bob takes the second qubit.

Each of Alice and Bob now flips a coin and does the following: If they flip Tails, they directly measure their qubit in the standard basis; if they flip Heads, they first apply a Hadamard to their qubit and then they measure in the standard basis.

1. Prove the following statements:
   - If Alice flips T and Bob flips T, it's *possible* Alice and Bob measure 1, 1 respectively.
   - If Alice flips T and Bob flips H, it's *impossible* Alice and Bob measure 1, 0 respectively.
   - If Alice flips H and Bob flips T, it's *impossible* Alice and Bob measure 0, 1 respectively.
   - If Alice flips H and Bob flips H, it's *impossible* Alice and Bob measure 1, 1 respectively.
2. Hardy says the following: "Consider the situation before any coin flips or measurements happen. What are the possible outcomes that the qubits can produce?
   - Consider the scenario where Alice flips T. Since it's possible that Bob also flips T, by the first statement in the previous part, it's *possible* for Alice to measure 1. Because of this, it must be *impossible* for Bob's qubit to produce a 0 when he flips Heads.
   - Now repeat the analysis but considering Bob's coin flip first. It is *possible* for Bob to see a 1 if he sees T, and in the case that he does, it is *impossible* for Alice to see a 0 if she sees H.
   - We thus conclude, that in case of flipping H, it is *impossible* for both Alice and Bob to see 0 respectively. Thus, they must both register a 1 in this case. However, this contradicts the fourth statement."

   Is the above critique a valid paradox? Do you agree or disagree with Hardy?

※ Problem 3: Error Reduction for BPP and BQP

Recall the bounded-error probabilistic classes we saw in class. Part of their definitions can be summarized as follows:

- (Completeness) If the input string x is in the language, accept with probability $2/3$.
- (Soundness) If the input string x is not in the language, accept with probability at most $1/3$.

These constants $2/3$ and $1/3$ may seem arbitrary, and you'd be correct! It doesn't really matter what these constants are, because we can simply repeat the algorithms and take a majority vote to boost the probabilities.

1. Suppose we had an algorithm for deciding a language in BQP. If we ran this algorithm three times and took the majority vote, what is the probability that it accepts a string x that is in the language?

2. What about the probability that it accepts a string that is **not** in the language?

3. Let $S_m = \sum_{i=1}^{m} X_i$, where X_i is a random number that takes either 0 or 1, and has probability $1/2 + \epsilon$ to have the value 1. We define the one sided Chernoff bound for this distribution as follows:

$$\Pr(\sum_{i=1}^{m} X_i < m/2) \leq e^{-2\epsilon^2 m}. \tag{3}$$

   Using the above bound, how many times do we need to repeat a BQP circuit to get the completeness (acceptance probability of a valid string) to at least $1 - 2^{-n}$?

※ Problem 4: Toffoli Gates for Simulating Classical Computation

To show that $P \subseteq BQP$, we must show that quantum computers can simulate classical computation. One bottleneck in simulating classical circuits directly, is that these circuits could be irreversible, which is not possible on a quantum computer. To reconcile this, we need to implement classical gates in a reversible way on a quantum computer.

The Toffoli gate takes in three input bits (a, b, c) and negates the third bit c if and only if $a \wedge b = 1$.

Show how to use the Toffoli gate to create a reversible AND gate and a reversible NOT gate.

❊ Problem 5 (Coding): Simulating the CHSH game

In class, we saw that the Bell inequality for the CHSH game states that the win rate for two parties with access to only classical information is at most 75%. However, if we give each of the players one qubit from a Bell pair, they can increase their win rate up to 85%.

1. Write some code to play the CHSH game, where Alice and Bob choose their answer bits randomly. How many games out of 1000 did they win?
2. How many games did they win out of 1000 where Charlie always give them both the bit 0?
3. Write code to play the CHSH game, where Alice and Bob share a pair of entangled qubits. Simulate their strategy based on receiving a qubit from Charlie. How many games out of 1000 did they win?
4. How many games did they win out of 1000 where Charlie always give them both the bit 0, but they are allowed to share an entangled pair of qubits?

❊ Bonus Reading

- **It's hard to think when someone Hadamards your brain** https://scottaaronson.blog/?p=3975
  This blog post by Scott Aaronson discusses Hardy's paradox and other similar thought experiments in more depth. One interesting reason these paradoxes are interesting, is that in principle our minds are also made up of quantum particles, so there is a way that they can be described using quantum mechanics. This paradox around causality has some implications on the way we understand how our perception of the universe works.