※ Information 2: Quantum Teleportation and Superdense Coding

"Information is the resolution of uncertainty."

- Claude Shannon

In this section, we will study another fundamental tool in quantum information, quantum teleportation. Teleportation is referring to the teleportation of information, meaning that we are transmitting information instantly. However, recall the caveat from our discussion from section 5.4 about spooky action at a distance.

Suppose Alice has a qubit in some unknown state

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{8}$$

and would like to send this state to Bob. In preparation for the teleportation protocol, Alice and Bob each took one qubit from a singlet state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ which they generated in advance.

Question 15. Is it possible to have a teleportation protocol where Alice keeps her state $|\psi\rangle$ *and* sends the state to Bob?

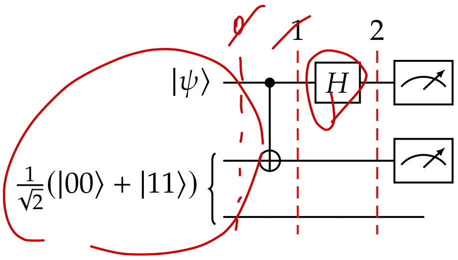
No! That would violate the no cloning theorem.

The following is the circuit for quantum teleportation.

$|\psi\rangle \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Alice

Bob



$\rightarrow |\psi\rangle$.

Question 16. What is the state of the system after each step?

0: $(\alpha|0\rangle + \beta|1\rangle) \otimes \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$

1: $(CNOT_{1,2})\frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle)$

$= \frac{1}{\sqrt{2}}(\alpha\, CNOT_{1,2}|000\rangle + \alpha\, CNOT_{1,2}|011\rangle + \beta\, CNOT_{1,2}|100\rangle + \beta\, CNOT_{1,2}|111\rangle)$

$= \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle)$

2: (H gate on qubit 1).

$\frac{1}{\sqrt{2}}(\alpha \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |00\rangle + \alpha \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |11\rangle + \beta \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |10\rangle + \beta \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |01\rangle)$

$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}(\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle + \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle)$

The state before measurement:

Question 17. Write down Bob's state after Alice measures her qubits in the standard basis, for each output she sees.

- $|00\rangle$: $\dfrac{\frac{1}{2}(\alpha|000\rangle + \beta|001\rangle)}{\sqrt{|\alpha/2|^2 + |\beta/2|^2}} = \alpha|000\rangle + \beta|001\rangle = |00\rangle \otimes \boxed{(\alpha|0\rangle + \beta|1\rangle)}$

  $\rightarrow \sqrt{\tfrac{1}{4}(|\alpha|^2 + |\beta|^2)} = \tfrac{1}{2}\sqrt{|\alpha|^2 + |\beta|^2}$
- $|01\rangle$: $\alpha|1\rangle + \beta|0\rangle$.
- $|10\rangle$: $\alpha|0\rangle - \beta|1\rangle$
- $|11\rangle$: $\alpha|1\rangle - \beta|0\rangle$

Question 18. For each of the resulting states, what gates can Bob apply to return to $|\psi\rangle$?

$|00\rangle \rightarrow I$ $\qquad$ $|10\rangle \rightarrow Z$

$|01\rangle \rightarrow X$ $\qquad$ $|11\rangle \rightarrow XZ$.

At the end of the algorithm, Alice's initial qubit is destroyed and the state has been successfully "teleported" to Bob. At the end, we will have one of the following four final states.

- $|00\rangle \otimes |\psi\rangle$
- $|01\rangle \otimes |\psi\rangle$
- $|10\rangle \otimes |\psi\rangle$
- $|11\rangle \otimes |\psi\rangle$

This protocol even works when the state to be teleported is entangled with other qubits! The consequence of this is that if we have some entangled state $|\phi\rangle$ over n qubits, the full state can be teleported after repeating the procedure n times.

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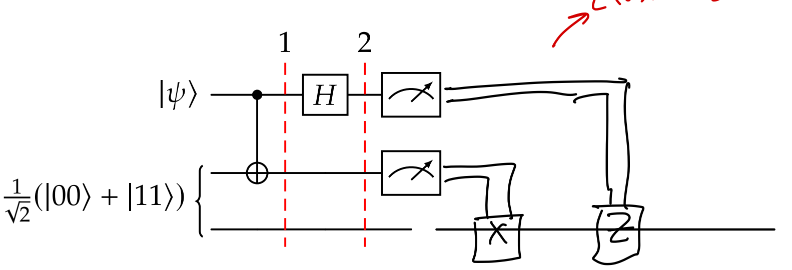
The following is the circuit for quantum teleportation.

classically controlled



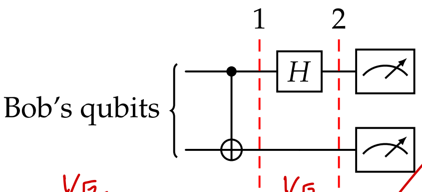
Question 16. What is the state of the system after each step?

7.1 Superdense Coding

Suppose Alice wants to send 2 bits of classical information to Bob. We will see here that there is a way to communicate these bits of classical information by just sending a single qubit. Again, Alice and Bob will prepare by creating an entangled pair of qubits $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Depending on the information Alice wants to send, she performs the following operations:

- "00": $I \rightarrow$ $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ $|\Phi^+\rangle$
- "01": $\cancel{Z}$ X $\rightarrow$ $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ $|\Phi^-\rangle$
- "10": $\cancel{X}$ Z $\rightarrow$ $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$ $|\Psi^+\rangle$
- "11": $ZX \rightarrow$ $\frac{1}{\sqrt{2}}(-|10\rangle + |01\rangle)$ $|\Psi^-\rangle$

Question 19. The following circuit outlines the superdense coding protocol. Use the table below to analyze the state of the system after each step, for each possible input.



$|-\rangle|0\rangle$

$|+\rangle|0\rangle$

$1/\sqrt{2}$ $1/\sqrt{2}$

Alice's Message	Bob's Qubits	After CNOT	After H
00	$\|00\rangle + \|11\rangle$	$\|00\rangle + \|10\rangle$	$\|00\rangle$
01	$\|00\rangle - \|11\rangle$	$\|00\rangle - \|10\rangle$	$\|10\rangle$
10	$\|10\rangle + \|01\rangle$	$\|11\rangle + \|01\rangle$	$\|01\rangle$
11	$-\|10\rangle + \|01\rangle$	$-\|11\rangle + \|01\rangle$	$\|11\rangle$

$|+\rangle|1\rangle$

$|-\rangle|1\rangle$

Table 1: Fill out the table above to see the result of applying each step of the circuit.