CS166 WI24: Midterm 1

※ Problem 1

For each of the following matrices, determine if they are stochastic or unitary. Some matrices may be both. If the matrix is unitary, write down the matrix that represents the inverse of its action.

1.1 $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

1.2 $\begin{bmatrix} 0 & -i \\ 1 & 0 \end{bmatrix}$

1.3 $\begin{bmatrix} 0 & 1/3 \\ 1 & 2/3 \end{bmatrix}$

1.4 $\frac{1}{\sqrt{2}} \begin{bmatrix} i & -1 \\ -i & -1 \end{bmatrix}$

1.5 $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 1/2 \end{bmatrix}$

※ Problem 2

For each of the following circuits, write down the probability of getting each outcome when measuring them in the specified basis. You may need the following gates:

$$X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{1}$$

$$H := \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \quad R_\theta := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad S := \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \quad T := \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix} \tag{2}$$

2.1 Measure in the standard basis

$$\begin{quantikz} \lstick{$|0\rangle$} & \gate{H} & \gate{Y} & \gate{Z} & \gate{Y} & \meter{} \end{quantikz}$$

2.2 Measure in the Hadamard basis

Recall that $\cos(\pi/4) = \sin(\pi/4) = \frac{1}{\sqrt{2}}$.

$$\begin{quantikz} \lstick{$|1\rangle$} & \gate{R_{\pi/4}} & \gate{S} & \gate{H} & \gate{X} & \meter{} \end{quantikz}$$

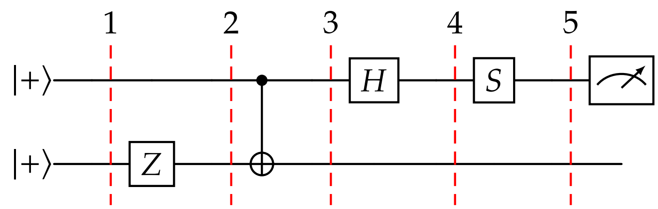
2.3 Measure in the Hadamard basis

$$\begin{quantikz} \lstick{$|0\rangle$} & \gate{Y} & \gate{Z} & \gate{H} & \gate{H} & \gate{Z} & \gate{Y} & \gate{H} & \gate{X} & \gate{X} & \gate{H} & \meter{} \end{quantikz}$$

※ Problem 3

3.1

Consider the following circuit. Write down the state of the system at each time step.



3.2

Let the state of the system before the measurement be $|\psi\rangle$. What is $\langle\psi|$?

3.3

If we measure the first qubit at the end of the above circuit in the standard basis and observe $|1\rangle$, what is the state after the measurement?

※ Problem 4

4.1

The set of all complex numbers with norm 1 can be written in the form $e^{i\theta}$. For example, the number 1 is when $\theta = 0$, and the number i is when $\theta = \pi/2$.

Draw $e^{i2\pi/5}$ in the complex plane, and write it in standard form.

4.2

What is the result of multiplying a complex number $\alpha_1 = e^{i\theta_1}$ with another complex number $\alpha_2 = e^{i\theta_2}$? Draw α_1, α_2, and the product $\alpha_1\alpha_2$ in the complex plane.

4.3

Find the fifth roots of unity. That is, a set of five complex numbers $\{1, \omega, \omega^2, \omega^3, \omega^4\}$ such that for any element ω^i in the set, we have

$$(\omega^i)^5 = 1. \tag{3}$$

4.4

Recall the Pauli Z gate defined as follows:

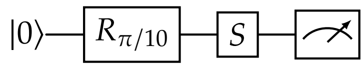
$$Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{4}$$

Find a 5-th root of the Z gate. That is, a gate P such that

$$P^5 = Z. \tag{5}$$

※ Problem 5

Consider the following circuit:



5.1

If we measure the above circuit in the standard basis, what are the possible outcomes and expected probabilities of seeing each?

5.2

If we sampled from the above circuit 1000 times, and found an estimate for the probability p of measuring $|0\rangle$, what is the probability that our estimate is off by more than 0.05 using the Chebyshev inequality?

5.3

If we sampled from the above circuit 1000 times, and found an estimate for the probability p of measuring $|0\rangle$, what is the probability that our estimate is off by more than 0.05 using the Chernoff bound?

5.4

We implemented the above circuit on some quantum system. After measuring the result in the standard basis a total of 1000 times, we got the following results:

$|0\rangle$: 257 times

$|1\rangle$: 743 times

Based on your analysis from the previous two parts, is this a reasonable outcome? If it is not, what is a possible explanation for what happened?