Module 1: Foundations 2

Contents

※ Foundations 2b: Single Qubit Systems (1/15)

"Truth is simple yet purposely complex."

— Wald Wassermann

A quantum algorithm comprises of three main components.

1. Storing quantum information (statevector)
2. Manipulating the stored quantum information (unitary transformations), and
3. Extracting a result (quantum measurement).

Let's formalize these notions and examine examples over small quantum systems.

4.1 Storing Quantum Information

The smallest quantum system we can define is a single qubit.

Definition 4.1 (Qubit)**.** A **qubit** is an object whose state can be represented by a unit vector $|\psi\rangle \in \mathbb{C}^2$.

As elements of a Hilbert space, any qubit should be expressible as a linear combination of some basis vectors. The first basis we will use is the standard basis.

Example 4.2. The **standard basis** of a single qubit is the set

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}. \tag{1}$$

In braket notation, we will describe these vectors as $|0\rangle := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $|1\rangle := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Example 4.3. The **Hadamard basis** is the set containing the following two vectors:

$$|+\rangle := \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |1\rangle\,, \tag{2}$$

$$|-\rangle := \frac{1}{\sqrt{2}} |0\rangle - \frac{1}{\sqrt{2}} |1\rangle\,. \tag{3}$$

Question 1. Verify that the Hadamard basis is indeed an orthonormal basis for a single qubit.

Check normality: $\langle +|+\rangle = \langle -|-\rangle = 1$

$\langle +|+\rangle = \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)\frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$= \frac{1}{2}(\underbrace{\langle 0|0\rangle}_{1} + \underbrace{\langle 0|1\rangle}_{0} + \underbrace{\langle 1|0\rangle}_{0} + \underbrace{\langle 1|1\rangle}_{1}) = 1.$ ✓

$|0\rangle$ is unit vector $|0\rangle$ and $|1\rangle$ are orthogonal.

Check orthogonality $\langle +|-\rangle = \langle -|+\rangle = 0.$

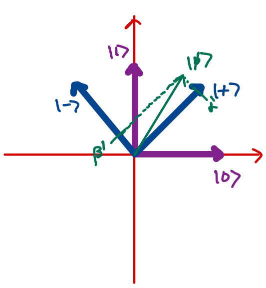
$\langle +|-\rangle = \frac{1}{\sqrt{2}}(\langle 0| + \langle 1|)\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$= \frac{1}{2}(\underbrace{\langle 0|0\rangle}_{1} - \underbrace{\langle 0|1\rangle}_{0} + \underbrace{\langle 1|0\rangle}_{0} - \underbrace{\langle 1|1\rangle}_{1}) = 0.$ ✓

Any state can be written in the Hadamard basis as well. This is equivalent to saying that there exist unique scalars α' and β' such that

$$|\psi\rangle := \alpha\,|0\rangle + \beta\,|1\rangle = \alpha'\,|+\rangle + \beta'\,|-\rangle\,. \tag{4}$$

Question 2. Let $|\phi\rangle := \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$. Express $|\phi\rangle$ in the Hadamard basis.



To find the length of the projection of $|\phi\rangle$ onto $|+\rangle$, we do $\langle +|\phi\rangle$.

To find the length of the projection of $|\phi\rangle$ onto $|-\rangle$, we do $\langle -|\phi\rangle$.

$\langle +|\phi\rangle = \left(\frac{1}{\sqrt{2}}\langle 0| + \frac{1}{\sqrt{2}}\langle 1|\right)\left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right) = \frac{1}{2\sqrt{2}}\underbrace{\langle 0|0\rangle}_{1} + \frac{\sqrt{3}}{2\sqrt{2}}\underbrace{\langle 0|1\rangle}_{0} + \frac{\sqrt{3}}{2\sqrt{2}}\underbrace{\langle 1|0\rangle}_{0} + \frac{\sqrt{3}}{2\sqrt{2}}\underbrace{\langle 1|1\rangle}_{1}$

$= \frac{1+\sqrt{3}}{2\sqrt{2}}.$

$\langle -|\phi\rangle = \left(\frac{1}{\sqrt{2}}\langle 0| - \frac{1}{\sqrt{2}}\langle 1|\right)\left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right) = \frac{1}{2\sqrt{2}}\langle 0|0\rangle - \frac{\sqrt{3}}{2\sqrt{2}}\langle 1|1\rangle = \frac{1-\sqrt{3}}{2\sqrt{2}}.$

$\Rightarrow |\phi\rangle = \left(\frac{1+\sqrt{3}}{2\sqrt{2}}\right)|+\rangle + \left(\frac{1-\sqrt{3}}{2\sqrt{2}}\right)|-\rangle.$

In the case of having more than one qubit, we can think of having a system that can be represented by a unit vector in $(\mathbb{C}^2)^{\otimes n}$, where the tensor power notation is shorthand for

$$(\mathbb{C}^2)^{\otimes n} := \mathbb{C}^2 \otimes \overset{n}{\cdots} \otimes \mathbb{C}^2 = \mathbb{C}^{2^n}. \tag{5}$$

Example 4.4 (3 qubit system)**.** The **standard basis** for 3 qubits is the set of all combinations of the tensor product over single qubit standard basis states. For example, one such state is

$$|000\rangle := |0\rangle \otimes |0\rangle \otimes |0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}. \tag{6}$$

We like to abbreviate $\otimes$ when using braket notation if it isn't too confusing. Using this shorthand, the full set of standard basis states for 3 qubits is

$$\{|000\rangle, |001\rangle, |010\rangle, |011\rangle, |100\rangle, |101\rangle, |110\rangle, |111\rangle\}. \tag{7}$$

We will also often write these states as

$$\{|0\rangle, |1\rangle, |2\rangle, |3\rangle, |4\rangle, |5\rangle, |6\rangle, |7\rangle\}. \tag{8}$$

4.2 Transforming Quantum Systems

A quantum state should be only transformed into another quantum state. By our definition, this means that the transformation should preserve the L2-norm, and it should also be linear. An operation that satisfies these properties is called **unitary**, and every unitary transformation can be represented by a **unitary matrix**. This is the family of "quantum gates" we will be using throughout this course.

Example 4.5. Suppose we have a system of 2 qubits in a quantum state:

$$|\phi\rangle = \begin{bmatrix} -i/2 \\ 1/2 \\ -1/2 \\ i/2 \end{bmatrix}, \tag{9}$$

and we have a 2 qubit unitary represented by the matrix

$$A = \begin{bmatrix} \frac{1}{2} & 0 & -\frac{1}{2} & 0 \\ 0 & \frac{1+i}{2} & 0 & \frac{-1-i}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1+i}{2} & 0 & \frac{i+1}{2} \end{bmatrix}. \tag{10}$$

Question 3. What is the state of our system after A is applied to $|\phi\rangle$?

$$\begin{pmatrix} 1/2 & 0 & -1/2 & 0 \\ 0 & 1+i/2 & 0 & -1-i/2 \\ 1/2 & 0 & 1/2 & 0 \\ 0 & 1+i/2 & 0 & 1+i/2 \end{pmatrix} \begin{pmatrix} -i/2 \\ 1/2 \\ -1/2 \\ i/2 \end{pmatrix} = \begin{pmatrix} -i/4 + 1/4 \\ 1+i/4 + 1+i/4 \\ -i/4 - 1/4 \\ 1+i/4 + -1+i/4 \end{pmatrix}$$

Definition 4.6 (Adjoint)**.** The **adjoint** $A^\dagger$ (read "A dagger") of a matrix A is the matrix you get after transposing A and taking the complex conjugate of each element. This process is also referred to as taking the **conjugate transpose** of a matrix.

Question 4. What is the adjoint of A?

$$A^\dagger = \begin{bmatrix} 1/2 & 0 & 1/2 & 0 \\ 0 & 1-i/2 & 0 & 1-i/2 \\ -1/2 & 0 & 1/2 & 0 \\ 0 & i-1/2 & 0 & 1-i/2 \end{bmatrix}$$

Question 5. Let $A = \begin{bmatrix} \frac{\sqrt{3}i}{2} & -\frac{1}{2} \\ \frac{i}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, and $|\phi\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$.

$$A^\dagger = \begin{bmatrix} -\frac{\sqrt{3}}{2}i & -\frac{i}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$$

1. Calculate $A|\phi\rangle$.

   $$A|\phi\rangle = \begin{bmatrix} \frac{\sqrt{3}i}{2} & -\frac{1}{2} \\ \frac{i}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}i + i}{2\sqrt{2}} \\ \frac{i - \sqrt{3}i}{2\sqrt{2}} \end{bmatrix}$$

adjoint of the vector!

2. Calculate $A^\dagger|\phi\rangle$.

   $$A^\dagger|\phi\rangle = \begin{bmatrix} -\frac{\sqrt{3}}{2}i & -\frac{i}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{-\sqrt{3}i - 1}{2\sqrt{2}} \\ \frac{-1 - \sqrt{3}i}{2\sqrt{2}} \end{bmatrix}$$

3. Calculate $\langle\phi|A^\dagger$.

   $$(A|\phi\rangle)^\dagger = \langle\phi|A^\dagger = \begin{bmatrix} \frac{-\sqrt{3}i - i}{2\sqrt{2}} & \frac{-i + \sqrt{3}i}{2\sqrt{2}} \end{bmatrix}$$

4. Calculate $\underbrace{\langle\phi|A^\dagger}_{\text{adjoint of the same state!}}\underbrace{A|\phi\rangle}_{\text{quantum state}}$. $\Rightarrow$ Norm of a quantum state is 1.

One of the most basic unitary matrices that will show up is the **identity matrix**. This is the matrix with ones on its diagonals and zeros everywhere else. We will use I_n to denote the $n \times n$ identity matrix.

Let's see a consequence of requiring an operator U to be norm-preserving when applied to a quantum state. We can write this condition mathematically as

$$|| \, |\psi\rangle \, ||_2 = 1 \to ||U \, |\psi\rangle \, ||_2 = 1, \tag{11}$$

which we know can alternatively be written as

The only way this is true is if $U^\dagger U = I$.

$$\langle\psi|\psi\rangle = 1 \rightarrow \langle\psi|U^\dagger U|\psi\rangle = 1. \tag{12}$$

An important consequence of this fact is that all quantum operations are **reversible**. For any starting state $|\psi\rangle$ and norm preserving operation U, we can find the inverse operator $U^\dagger$ which is simply the adjoint of U.

Question 6. Are classical operations reversible? If not, describe an operation that is not reversible.

No!

e.g.) Max of an array   max([0, 1, 2, 3]) → 3   can't reverse...

e.g.) Reset to zero   7 → 0

Definition 4.7. A matrix U is a **unitary matrix** if any of the following equivalent definitions are true.

1 • U is norm-preserving.

2 • U preserves inner products (inner product of $|\psi\rangle$ and $|\phi\rangle$ is equal to the inner product of $U|\psi\rangle$ and $U|\phi\rangle$.

3 • $U^\dagger U = UU^\dagger = I$.

4 • Rows of U form an orthonormal basis.

5 • Columns of U form an orthonormal basis.

Question 7. Prove that all of the above definitions for a unitary matrix are equivalent.

3 ⇒ 2 $\langle\phi|U^\dagger U|\psi\rangle = \langle\phi|\psi\rangle$ ✓.

4.3 Single Qubit Operations

Here we will look at examples of important single qubit operators.

Question 8. Determine the action of the following single qubit operators on the standard basis states. What about their action on a state in superposition, $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$?

$$1)\ I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad 2)\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad 3)\ \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} \quad 4)\ \begin{bmatrix} 0 & -i \\ -1 & 0 \end{bmatrix} \quad 5)\ H = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{13}$$

① $I|\psi\rangle = |\psi\rangle$.

② $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$

③ $\begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ i\beta \end{bmatrix}$

④ $\begin{bmatrix} 0 & -i \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} -i\beta \\ -\alpha \end{bmatrix}$

⑤ $\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}}(\alpha+\beta) \\ \frac{1}{\sqrt{2}}(\alpha-\beta) \end{bmatrix}$

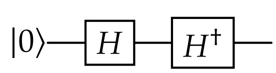
The final operator we looked at in the previous question is called the **Hadamard gate** H. It serves the important function of transitioning between the standard and Hadamard basis. It also is a good gate for us to see our first example of **negative interference**.

Question 9. What is the adjoint of the Hadamard gate? Verify this by computing $H^\dagger H|0\rangle$.

$H^\dagger = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = H$.

$H^\dagger H = \left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right)\left(\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = I$ ✓.

Let's use the previous question to see how we draw quantum circuits.



The starting state of each qubit is written on the left, and the sequence of gates we apply goes from left to right. For reasons that we will study later, you never increase the number of qubits you start with.

Definition 4.8 (Important Single Qubit Gates)**.** The following gates will appear frequently throughout this course and are standard in the literature.

- Hadamard gate: $H := \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.
- Pauli gates:

$$X := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad Y := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \quad Z := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{14}$$

$$YH|0\rangle = Y\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = Y\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} -i/\sqrt{2} \\ i/\sqrt{2} \end{bmatrix}$$

Question 10. What is the state of the qubit at the end of this circuit?

$$|0\rangle — \boxed{H} — \boxed{Y} —$$

Remember that since a qubit is a unit vector, if we know it has real amplitudes, it is a vector on the unit circle of the plane. This motivates the following standard gate we will be using as well.

Definition 4.9 (Rotation gate)**.** The rotation gate R_θ is a parameterized gate defined as

$$R_\theta := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{15}$$

$$R_\theta^\dagger R_\theta = \begin{bmatrix} c\theta & s\theta \\ -s\theta & c\theta \end{bmatrix}\begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix} = \begin{bmatrix} c^2+s^2 & -cs+cs \\ -cs+cs & s^2+c^2 \end{bmatrix} = I \checkmark$$

Question 11. Verify that the rotation gate is indeed unitary. (Abbreviated $\cos\theta$ as $c\theta$ and $\sin\theta$ as $s\theta$).

Question 12. Compute the following: $R_{\pi/4} = \begin{bmatrix} \cos\pi/4 & -\sin\pi/4 \\ \sin\pi/4 & \cos\pi/4 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$

- $R_{\pi/4}|0\rangle$ $\quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = |+\rangle.$
- $R_{\pi_4}|+\rangle$ $\quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = |1\rangle$
- $R_{\pi_4}|1\rangle$ $\quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = -|-\rangle = e^{i\pi}|-\rangle \equiv |-\rangle.$ global phase!
- $R_{\pi_4}|-\rangle$ $\quad \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = |0\rangle.$

You may need the fact that, $\cos(\pi/4) = \sin(\pi/4) = \frac{1}{\sqrt{2}}$.

Let's now look at the full family of single qubit states where we allow complex amplitudes. In your homework, you observed that multiplying a state by a **global phase** does not change the probability of observing an outcome. That is, multiplying both amplitudes by the same complex number $e^{i\theta}$ didn't change the measurement probabilities.

Things get more interesting when we consider **relative phases**. These are phases that are applied separately to each amplitude, through transformations that might look like

$$\alpha|0\rangle + \beta|1\rangle \rightarrow \alpha|0\rangle + \beta e^{i\theta}|1\rangle . \tag{16}$$

Question 13. The Hadamard basis states can be written as two states that differ by a relative phase:

$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \tag{17}$$

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle + e^{i\pi}\frac{1}{\sqrt{2}}|1\rangle . \tag{18}$$

What is the probability of measuring $|0\rangle$ and $|1\rangle$ for each of these states? What if we apply the H gate first to each state and then measure?

Prob. of measuring for $|+\rangle$.

- $|0\rangle \rightarrow |\langle 0|+\rangle|^2 = |1/\sqrt{2}|^2 = 1/2$
- $|1\rangle \rightarrow |\langle 1|+\rangle|^2 = 1/2$

Prob. of measuring for $|-\rangle$.

- $|0\rangle \rightarrow |\langle 0|-\rangle|^2 = |1/\sqrt{2}|^2 = 1/2$

Apply $\boxed{H}$.

$H|+\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix} = |0\rangle.$

$H|-\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}\begin{bmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix} = |1\rangle.$

The above question highlights that relative phase is different from global phase, and can actually be detected. It seems then, that the most general way to express the state of a qubit is something like the following:

$$|\psi\rangle := m_0 e^{i\pi\phi_0}|0\rangle + m_1 e^{i\pi\phi_1}|1\rangle . \tag{19}$$

Thus we have found a four variable parameterization of a qubit. However, we only have two real constraints for a qubit state:

1. The L_2-norm is 1.
2. The global phase does not matter.

Question 14. Use the constraint of the $L2$ norm to express m_0 and m_1 using a single parameter.

Let $m_0 = \cos\theta$, $m_1 = \sin\theta$. Then,

$|\psi\rangle = \cos\theta\, e^{i\pi\phi_0}|0\rangle + \sin\theta\, e^{i\pi\phi_1}|1\rangle.$

We can express both using θ.

Question 15. Use the constraint on the global phase factor to express ϕ_0 and ϕ_1 using a single parameter.

Multiply the whole state by $e^{i\pi(-\phi_0)}$. Then,

$$e^{i\pi(-\phi_0)}|\psi\rangle = \cos\theta \underbrace{e^{i\pi(\phi_0-\phi_0)}}_{1.} |0\rangle + \sin\theta\, e^{i\pi(\phi_1-\phi_0)} = \cos\theta\,|0\rangle + \sin\theta\, e^{i\pi(\phi_1-\phi_0)}.$$

Let $\phi := \phi_1 - \phi_0$. Then we can use a single variable.

Lemma 4.10 (Parameterization of a qubit)**.**

for $\theta, \phi \in [0, 2\pi)$

$$\boxed{|\psi\rangle = \cos\theta\,|0\rangle + \sin e^{i\pi\phi}\,|1\rangle.}$$

Hopefully you have a better grasp on what a quantum operation looks like. Most critically, we can sum up the important properties of quantum operations as follows:

1. **Reversible.** By unitarity, any operation U has a reverse operation $U^\dagger$.
2. **Deterministic.** The action of a unitary U is well defined.
3. **Continuous.** Any unitary U can be performed "half-way".

Question 16. Find the "half-way" unitary operator of Z. That is, find the unitary U such that $U \cdot U = Z$.

$$U = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}.$$

4.4 Single Qubit Measurements

The final piece of quantum algorithms we need to formalize are quantum measurements. In contrast to the properties that transformations had, quantum measurements have the exact opposite properties.

1. **Irreversible**. Once a measurement is performed, there is no way to recover the state before without rerunning the entire algorithm.
2. **Probabilistic**. The probability of observing an outcome is the squared norm of the amplitude corresponding to that outcome.
3. **Discrete**. Measurement results are inherently discrete.

Up until know, all measurements I described were performed in the **standard basis**. However, it turns out that we can measure in any basis we like! If you fix a basis, every qubit has a unique representation in that basis.

Example 4.11. Consider the Hadamard basis $|+\rangle$, $|-\rangle$. Suppose we have a state $|\psi\rangle := \alpha|0\rangle + \beta|1\rangle$. We can express this state in the Hadamard basis with another pair of amplitudes α' and β' as

$$|\psi\rangle = \alpha'|+\rangle + \beta'|-\rangle\,. \tag{20}$$

Measuring in a different basis affects the measurement outcomes. If we measure in the standard basis, $\{|0\rangle, |1\rangle\}$:

1. $|0\rangle$ with probability $|\langle 0|\psi\rangle|^2 = |\alpha|^2$, and
2. $|1\rangle$ with probability $|\langle 1|\psi\rangle|^2 = |\beta|^2$.

If we measure in the ~~standard basis, $\{|0\rangle, |1\rangle\}$:~~ Hadamard basis $\{|+\rangle, |-\rangle\}$

1. $|+\rangle$ with probability $|\langle +|\psi\rangle|^2 = |\alpha'|^2$, and
2. $|-\rangle$ with probability $|\langle -|\psi\rangle|^2 = |\beta'|^2$.

Question 17. What are the outcomes and corresponding probabilities if we measure $|1\rangle$ in the standard basis? What about in the Hadamard basis?

Standard basis:

$|0\rangle$ w.p. $|\langle 0|1\rangle|^2 = 0$

$|1\rangle$ w.p. $|\langle 1|1\rangle|^2 = 1$

Hadamard basis:

$|+\rangle$ w.p. $|\langle +|1\rangle|^2 = |\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$

$|-\rangle$ w.p. $|\langle -|1\rangle|^2 = |-\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$

The standard basis and Hadamard basis are a pair of **complementary bases**. This means that if you are certain that the state is in $|+\rangle$ or $|-\rangle$, you will be maximally *uncertain* of the state in the standard basis, and vice versa. Another way to think of unitary matrices is that they are a transformation between two orthonormal bases. For example, the Hadamard basis is a transformation between the standard and Hadamard bases.

$|\psi\rangle$ — $\boxed{H^\dagger}$ — $\boxed{\measuredangle}$ Hadamard basis measurement

$$|+\rangle \overset{H}{\leftrightarrow} |0\rangle \tag{21}$$

$$|-\rangle \overset{H}{\leftrightarrow} |1\rangle \tag{22}$$

$\underbrace{\langle 0|H^\dagger}|\psi\rangle = \langle +|\psi\rangle$

A useful consequence of this is that if we want to simulate a measurement in the Hadamard basis, we can apply the Hadamard gate and then measure in the standard basis. Let's now define measurements formally.

Definition 4.12 (Measurements)**.** Let $|\psi\rangle \in \mathbb{C}^N$ be a quantum state on n qubits and $\{|v_1\rangle, \ldots, |v_N\rangle\}$ be an orthonormal basis. If we measure in this basis, we will see outcome $|v_i\rangle$ with probability $|\langle v_i|\psi\rangle|^2$.

4.5 Distinguishing non-orthogonal states

Sometimes, we are interested in distinguishing outcomes of an experiment where the possible results are not necessarily orthogonal. This arises frequently in quantum computing with our limited capabilities of combating noise.

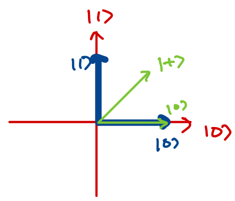
Consider the following experiment. We are running a simulation on a quantum device, which in theory should measure $|0\rangle$ with probability 1. However, there is some external noise affecting our system such that $1/2$ of the time the experiment will end in the state $|+\rangle$ instead. In this case, we will say the experiment "failed". Can we effectively distinguish a failed experiment from a successful experiment?

Question 18. What are the probabilities and outcomes when we measure a successful vs. failed experiment in the standard basis?

Success: Output state is $|0\rangle$.

$|0\rangle$ w.p. $|\langle 0|0\rangle|^2 = 1$.

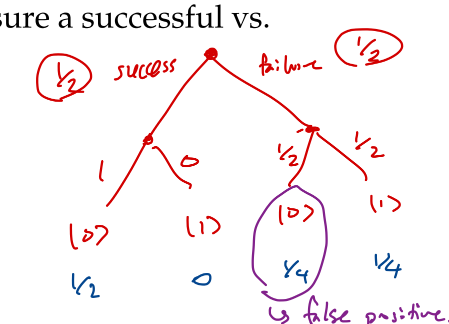
$|1\rangle$ w.p. $|\langle 1|0\rangle|^2 = 0$.



Failure: Output state is $|+\rangle$.

$|0\rangle$ w.p. $|\langle 0|+\rangle|^2 = |\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$

$|1\rangle$ w.p. $|\langle 1|+\rangle|^2 = |-\frac{1}{\sqrt{2}}|^2 = \frac{1}{2}$

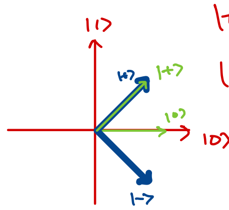


Question 19. What are the probabilities and outcomes when we measure a successful vs. failed experiment in the Hadamard basis?

Success: Output state is $|0\rangle$.

$|+\rangle$ w.p. $|\langle +|0\rangle|^2 = \frac{1}{2}$

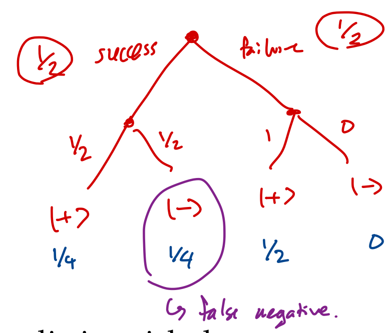
$|-\rangle$ w.p. $|\langle -|0\rangle|^2 = \frac{1}{2}$



Failure: Output state is $|+\rangle$.

$|+\rangle$ w.p. $|\langle +|+\rangle|^2 = 1$

$|-\rangle$ w.p. $|\langle -|+\rangle|^2 = 0$.



Question 20. There is no measurement that can be performed which can distinguish these outcomes perfectly. Is there a basis we can measure in that will do better than the previous two bases?

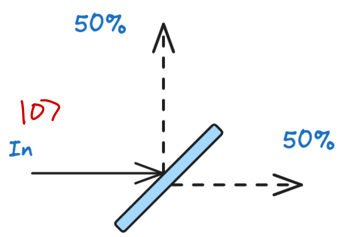
HW. Try to express orthonormal vectors using $\cos\theta$, $\sin\theta$.

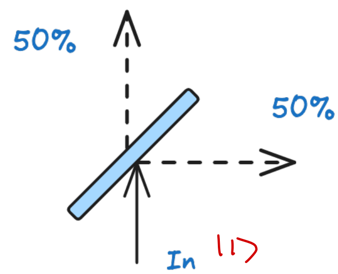
4.6 Beam Splitters

A **beam splitter** is a device which is used to split of beam of light into two. This works even if a single photon is fired, and the measurement statistics will be correctly reproduced! If we place a photon detector at the two possible directions the photon goes after it is split, we will detect the photon with probability 1/2 at each location. The same thing happens if we fire it from the perpendicular direction!

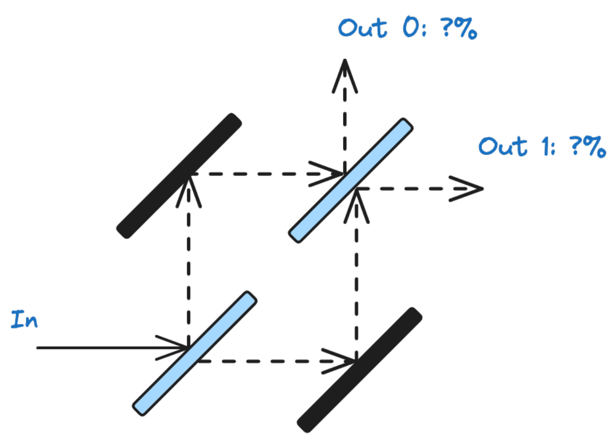
$|0\rangle$ is a right beam

$|1\rangle$ is a vertical beam





Question 21. What would you expect the outcome of the following setup to be? That is, with what probabilities will each of the detectors find the photons?

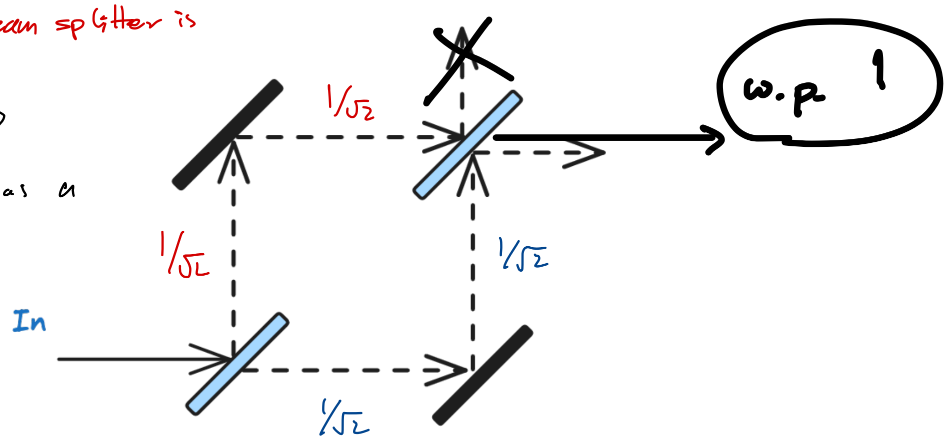


It turns out that the beam splitter acts like a Hadamard gate, and the horizontal "input" is like a $|0\rangle$ state, and the vertical "input" is like a $|1\rangle$ state.

State entering second beam splitter is

$\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = |+\rangle$

We model the beam splitter as a H gate on this state.



$H\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$

$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right) + \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle\right) = \left(\frac{1}{2}+\frac{1}{2}\right)|0\rangle + \left(\frac{1}{2}-\frac{1}{2}\right)|1\rangle.$

$= |0\rangle.$

4.7 Elitzur-Vaidman Bombs

Consider the following thought experiment. Suppose that we had a bomb that we can fire a photon through which has the following properties:

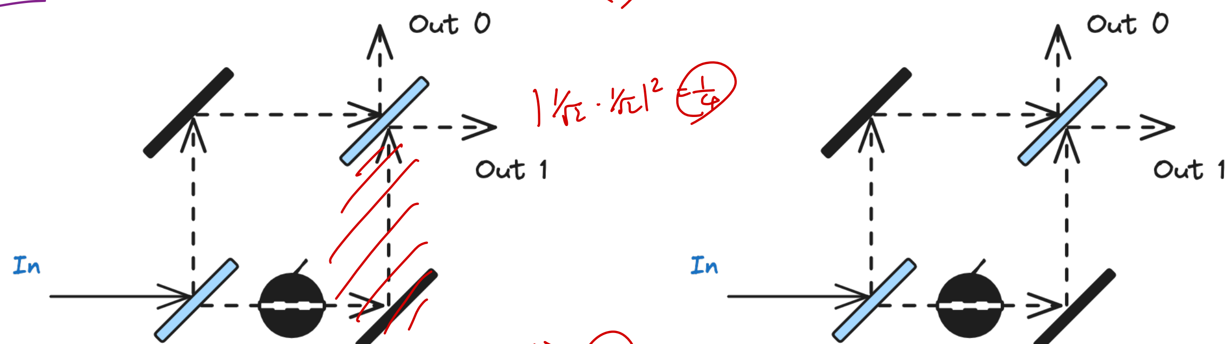
- If the bomb is not defective, the photon gets detected, and the bomb explodes.
- If the bomb is defective, the photon passes undetected, and the bomb does not explode.

Suppose we have many of these bombs and would like to find one which is not defective without exploding it. This is impossible classically, that is, if we don't take advantage of the quantum nature of light.

Defective case: Same as above figure!

Not defective

$|1/\sqrt{2} \cdot 1/\sqrt{2}|^2 = \frac{1}{4}$



• Out 1 w.p. 1

↳ explodes w.p. $|1/\sqrt{2}|^2 = \frac{1}{2}$

Question 22. What is the probability of observing each outcome if the bomb is defective?

Question 23. What is the probability the bomb explodes if the bomb is not defective? What is the probability of observing outcomes 0 and 1 if the bomb is not defective and it doesn't explode?

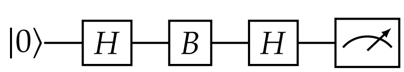
We can also describe this scenario using a quantum circuit, and creating a special gate to represent the bomb. We will define the gate B as follows:

- If the bomb is defective, it is just an identity gate I.
- If the bomb is not defective, measure in the standard basis. If $|0\rangle$ is observed, keep going. If $|1\rangle$ is observed, the bomb explodes.

Not defective.

$|0\rangle$ — [H] — [H] — [measure]

$= |0\rangle$ — [measure]



Defective

1/2 1/2

explosion

input to second splitter is $|0\rangle$.

1/2 1/2

$|0\rangle$ $|1\rangle$.

※ Foundations 2c: Two Qubits, Tensor Products, and Entanglement

Entanglement is not one but rather the characteristic trait of quantum mechanics.

- Erwin Schrodinger

The final building block for quantum systems that we need before going forward is **entanglement**. In simple terms, entanglement captures how dependent the probability of seeing one state is to the probability of seeing another state. It is a quantum generalization of joint and conditional distributions.

5.1 Partial Measurements

Once we have multiple qubits, we can decide to just measure one qubit out of the full system. Suppose Alice and Bob each have one qubit, and an operation is performed on the full system (both qubits). Is there something we can say about Bob's qubit by **only** measuring Alice's qubit?

We will use the following state to illustrate examples in this section:

$$|\psi\rangle := \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \alpha\,|00\rangle + \beta\,|01\rangle + \gamma\,|10\rangle + \delta\,|11\rangle\,. \tag{23}$$

We will say the first qubit is Alice's.

Question 24. If Alice performs a measurement in the standard basis, what is the probability that she observes $|0\rangle$? What is the state after the measurement? Can the state be written as a tensor product of two states?

Probabilities

$|0\rangle : |\alpha|^2 + |\beta|^2$

$|1\rangle : |\gamma|^2 + |\delta|^2$

If she observes $|0\rangle$,

$$\frac{\alpha|00\rangle + \beta|01\rangle}{\sqrt{|\alpha|^2 + |\beta|^2}} = |0\rangle \otimes \frac{\alpha|0\rangle + \beta|1\rangle}{\sqrt{|\alpha|^2+|\beta|^2}}.$$

The state collapses to the superposition of consistent states, then we must divide by the square root of the probability to normalize it.

Definition 5.1 (Partial Measurement Rule)**.** A quantum system collapses to fit the observed outcome from the measurement made.

Question 25. Write the state $|+-\rangle := |+\rangle \otimes |-\rangle$ in the standard basis.

$$|+-\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$
$$= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$$

Question 26. Consider the following two states.

- $|\phi_1\rangle := \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle$
- $|\phi_2\rangle := \frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle$

Express $|\phi_1\rangle \otimes |\phi_2\rangle$ in the standard basis.

$$|\phi_1\rangle \otimes |\phi_2\rangle = \left(\frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle\right) \otimes \left(\frac{1}{2}|0\rangle - \frac{\sqrt{3}}{2}|1\rangle\right)$$
$$= \frac{1}{4}|00\rangle - \frac{\sqrt{3}}{4}|01\rangle + \frac{\sqrt{3}}{4}|10\rangle - \frac{3}{4}|11\rangle$$

If a 2 qubit state can be written as a tensor product, we call the state **independent** or **separable**. Not every two qubit state is separable, with an important example being the following state.

$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}. \tag{24}$$

This state is so important in quantum information, that you'll see it referred to by different names including the **singlet state**, **Bell pair**, and others. When a state cannot be written as a tensor product of single qubit states, we say that the state is **entangled.**

$\begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}$

Question 27. If we measure the first qubit in a Bell pair, what is the probability of observing $|0\rangle$? What is the state of our system afterwards?

$\alpha = \frac{1}{\sqrt{2}}, \beta = 0, \gamma = 0, \delta = 1/\sqrt{2}.$

Probability of seeing $|0\rangle$ is $|\alpha|^2 + |\beta|^2 = \frac{1}{2}$.

The state after is $\frac{1/\sqrt{2}\,|00\rangle}{\sqrt{1/2}} = |00\rangle$

Question 28. Let $|\psi\rangle := \left(\frac{1}{\sqrt{3}}|0\rangle - \sqrt{\frac{2}{3}}|1\rangle\right) \otimes \left(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle\right)$. Suppose we measure the first qubit and observed $|0\rangle$. What was the probability of this event occurring? What is the state of the system after?

Typo! Should be $1/\sqrt{3}$.

Let $|\phi\rangle := \cancel{\frac{1}{3}}|00\rangle + \frac{1}{\sqrt{3}}|10\rangle + \frac{1}{\sqrt{3}}|11\rangle$ and perform the same analysis on this state. .

For $|\psi\rangle = \left(\frac{1}{\sqrt{6}}|00\rangle + \frac{1}{\sqrt{6}}|01\rangle - \frac{1}{\sqrt{3}}|10\rangle - \frac{1}{\sqrt{3}}|11\rangle\right)$

Prob. of $|0\rangle$ in first qubit is $|\frac{1}{\sqrt{6}}|^2 + |\frac{1}{\sqrt{6}}|^2 = \frac{1}{3}$.

State after is $\frac{1/\sqrt{6}\,|00\rangle + 1/\sqrt{6}\,|01\rangle}{\sqrt{1/3}}$

$|\phi\rangle = \frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|10\rangle + \frac{1}{\sqrt{3}}|11\rangle$

Prob. of $|0\rangle$ in first qubit is $|\frac{1}{\sqrt{3}}|^2 + |0|^2 = \frac{1}{3}$

State after is $\frac{1/\sqrt{3}\,|00\rangle}{\sqrt{1/3}} = |00\rangle$

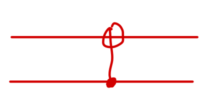
5.2 Two Qubit Operations

We will now define operations on two qubit states. Since the space is now 4 dimensions, these actions are represented by 4×4 unitaries.

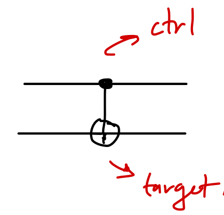
Definition 5.2 (CNOT gate)**.** The CNOT (Controlled-NOT) gate will flip the second qubit if the first qubit is $|1\rangle$, and otherwise do nothing.

CNOT w/ second qubit as control.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} =$$



$$CNOT := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. = \tag{25}$$



w/ qubit 1 as ctrl.

What if we want to apply a gate to only one qubit? The tensor product formalism gives us a nice way to define this. Suppose we want to apply an X gate to just the second qubit. We can represent this action as applying an I gate to the first qubit.

$$I \otimes X = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 0 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ 0 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 1 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \tag{26}$$

In general, if we have a separable state $|\psi_1\rangle \otimes |\psi_2\rangle$ and two independent operators A and B, we can represent the action $A \otimes B$ as follows:

$$(A \otimes B) |\psi_1\rangle \otimes |\psi_2\rangle = A |\psi_1\rangle \otimes B |\psi_2\rangle . \tag{27}$$

Question 29. Use the above shortcut to compute the action of $H \otimes H$ on all four standard basis states.

$(H \otimes H)(|0\rangle \otimes |0\rangle) = H|0\rangle \otimes H|0\rangle = |++\rangle$.

$(H \otimes H)(|0\rangle \otimes |1\rangle) = H|0\rangle \otimes H|1\rangle = |+-\rangle$

$(H \otimes H)(|1\rangle \otimes |0\rangle) = H|1\rangle \otimes H|0\rangle = |-+\rangle$

$(H \otimes H)(|1\rangle \otimes |1\rangle) = H|1\rangle \otimes H|1\rangle = |--\rangle$

Sometimes it will be easier to work in bra-ket notation rather than using matrices, especially as the size of the system increases. Consider the operation $I \otimes I \otimes H \otimes I$ on a system of 4 qubits. The matrix needed to represent this has a size of 16×16. Instead, we can simply analyze the effect it has on states that are relevant:

$$I_1 \otimes I_2 \otimes H_3 \otimes I_4 |0000\rangle = |00+0\rangle \tag{28}$$

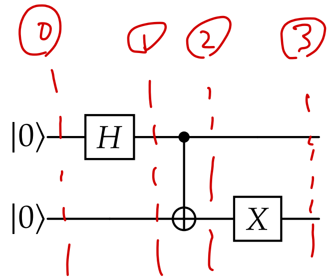
$$= |00\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |0\rangle \tag{29}$$

$$= \frac{1}{\sqrt{2}}(|0000\rangle + |0010\rangle). \tag{30}$$

By linearity of operators, we can just perform the analysis on each term independently before combining them together at the end.

5.3 Qubits in Quantum Circuit Notation

Consider the following two qubit circuit.



$= (I \otimes X)(CNOT)(H \otimes I)\,|00\rangle$

The circuit starts in the state $|00\rangle$. We then apply an H gate to just the first qubit, and then a CNOT gate with the first qubit as the control (a black circle represents the control bit).

Question 30. What is the state of the system after each operation?

0 $\quad |00\rangle$.

1 $\quad (H \otimes I)\,|00\rangle = H|0\rangle \otimes I\,|0\rangle$
$= |+\rangle \otimes |0\rangle = |+0\rangle$

2 $\quad CNOT\,|+0\rangle = CNOT\left(\tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|10\rangle\right)$
$= \tfrac{1}{\sqrt{2}}(CNOT)|00\rangle + \tfrac{1}{\sqrt{2}}(CNOT)\,|10\rangle$
$= \tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|11\rangle$.

3 $\quad (I \otimes X)\left(\tfrac{1}{\sqrt{2}}|00\rangle + \tfrac{1}{\sqrt{2}}|11\rangle\right)$ flip the second qubit.
$= \tfrac{1}{\sqrt{2}}(I \otimes X)|00\rangle + \tfrac{1}{\sqrt{2}}(I \otimes X)|11\rangle$
$= \tfrac{1}{\sqrt{2}}|01\rangle + \tfrac{1}{\sqrt{2}}|10\rangle$

5.4 Spooky Action at a Distance

Einstein never fully accepted quantum mechanics, in part because of the mysterious effects of entanglement. Once a Bell pair is created, in theory the corresponding qubits can be physically separated infinitely far. Experimentally, this has been verified up to 1000 miles apart!

$\tfrac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$

Why did this fact not sit well with Einstein and other physicists of the time? Suppose we created a Bell pair and moved one qubit to a different galaxy. If Alice now measures her qubit, Bob's state instantly collapses to the same state! This happens faster than the speed of light and Alice has no way of communicating her result to Bob. How does Bob's state know what to collapse to? Is it somehow "predetermined" by the two states?

Question 31. Suppose Alice and Bob decide to measure in the Hadamard basis instead. What is the probability that Alice observes $|+\rangle$, and what is Bob's state afterwards?

See next page.

Scientists referred to this and similar effects as "hidden variable theory", claiming that somehow the states agree upon how to be measured. We will later prove that there is no hidden variables, and the entanglement is really happening in the way we modeled!

$$\frac{1}{\sqrt{2}}\left(|0\rangle|0\rangle + |1\rangle|1\rangle\right).$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)|0\rangle + \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)|1\rangle\right)$$

$$= \frac{1}{2}|+\rangle|0\rangle + \frac{1}{2}|-\rangle|0\rangle + \frac{1}{2}|+\rangle|1\rangle - \frac{1}{2}|-\rangle|1\rangle$$

$\alpha|00\rangle \quad \beta|01\rangle \quad \gamma|10\rangle \quad \delta|11\rangle$

$$|0\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle + |-\rangle\right)$$

$$= \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}\right)$$

$$= \frac{1}{2}\left(\begin{bmatrix}1+1\\1-1\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$$

$$|1\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle - |-\rangle\right)$$

If we measure $|+\rangle$ for alice,

Prob. : $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 = \frac{1}{2}$.

State after: $\dfrac{\frac{1}{2}\left(|+\rangle|0\rangle + |+\rangle|1\rangle\right)}{\sqrt{\frac{1}{2}}}$

$$= \frac{1}{\sqrt{2}}\left(|+\rangle|0\rangle + |+\rangle|1\rangle\right)$$

$$= |+\rangle \otimes \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$$

$$= |+\rangle|+\rangle.$$