※ Information 2: Quantum Teleportation and Superdense Coding

8.1 Quantum Teleportation

"Information is the resolution of uncertainty."

- Claude Shannon

In this section, we will study another fundamental tool in quantum information, quantum teleportation. Teleportation is referring to the teleportation of information, meaning that we are transmitting information instantly.

Suppose Alice has a qubit in some unknown state

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle \tag{60}$$

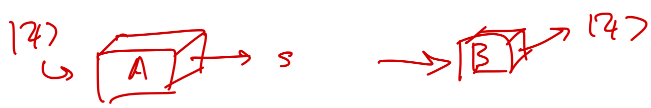
and would like to send this state to Bob. In preparation for the teleportation protocol, Alice and Bob each took one qubit from a singlet state $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ which they generated in advance.

Theorem 8.1. If Alice and Bob have a classical channel between them, but do not share any entanglement, then Alice cannot send an unknown quantum state to Bob.

If the state was known, Alice could send an approximation of the state by sending a classical string like "0.809017 |0> + 0.587785 |1>".

Contradiction

Pf/ Spse Alice could decompose an unknown q. state to a classical string, and Bob can use this string to create $|\psi\rangle$.



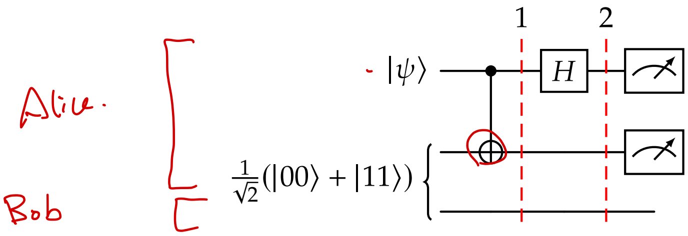
This creates a clone of $|\psi\rangle$! Violates No Cloning..

$\Rightarrow$ Alice cannot decompose unknown q. state to a string

Question 69. Is it possible to have a teleportation protocol where Alice keeps her state $|\psi\rangle$ *and* sends the state to Bob?

No! This violates No Cloning.

The following is the circuit for quantum teleportation.



Question 70. What is the state of the system after each step?

Init. $(\alpha|0\rangle + \beta|1\rangle)\left(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right) = \frac{1}{\sqrt{2}}\left(\alpha|000\rangle + \alpha|011\rangle + \beta|100\rangle + \beta|111\rangle\right)$

① $\frac{1}{\sqrt{2}}\left(\alpha|000\rangle + \alpha|011\rangle + \beta|110\rangle + \beta|101\rangle\right)$

② $\frac{1}{\sqrt{2}}\left(\alpha|{+}00\rangle + \alpha|{+}11\rangle + \beta|{-}10\rangle + \beta|{-}01\rangle\right)$

$= \frac{1}{2}\big(\alpha|000\rangle + \alpha|100\rangle + \alpha|011\rangle + \alpha|111\rangle$

$+ \beta|010\rangle - \beta|110\rangle + \beta|001\rangle - \beta|101\rangle\big)$

Question 71. Write down Bob's state after Alice measures her qubits in the standard basis, for each output she sees. Is there a gate that Bob can apply to his qubit to return the state to $|\psi\rangle$?

- $|00\rangle$: $\alpha|0\rangle + \beta|1\rangle$ $\Rightarrow$ I
- $|01\rangle$: $\alpha|1\rangle + \beta|0\rangle$ $\Rightarrow$ X
- $|10\rangle$: $\alpha|0\rangle - \beta|1\rangle$ $\Rightarrow$ Z
- $|11\rangle$: $\alpha|1\rangle - \beta|0\rangle$ $\Rightarrow$ XZ

~~**Question 72.** For each of the resulting states, what gates can Bob apply to return to $|\psi\rangle$?~~

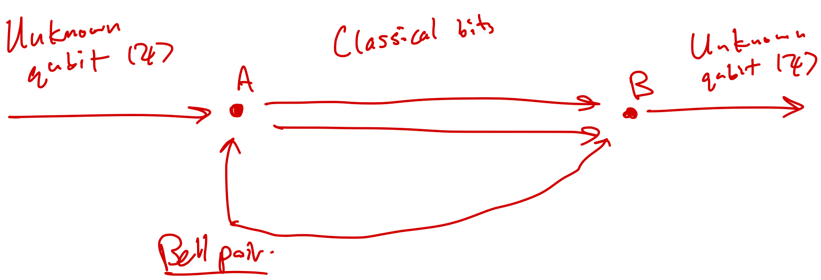
At the end of the algorithm, Alice's initial qubit is destroyed and the state has been successfully "teleported" to Bob. At the end, we will have one of the following four final states.

- $|00\rangle \otimes |\psi\rangle$
- $|01\rangle \otimes |\psi\rangle$
- $|10\rangle \otimes |\psi\rangle$
- $|11\rangle \otimes |\psi\rangle$

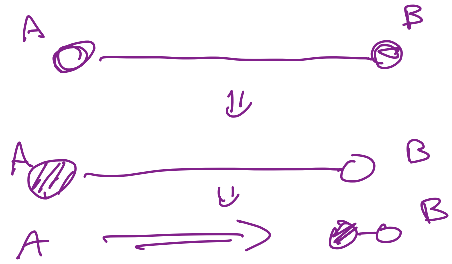
This protocol even works when the state to be teleported is entangled with other qubits! The consequence of this is that if we have some entangled state $|\phi\rangle$ over n qubits, the full state can be teleported after repeating the procedure n times.

One important caveat to this teleportation is that Bob needs to know what Alice's measurement result was! It turns out that faster than lightspeed communication cannot happen without some bonus verification procedure that is bound to lightspeed. This is why we are ok with a Bell pair collapsing faster than the speed of light, because if you don't **know** that it collapsed, it's as good as a state that hasn't collapsed.

The teleportation is still useful in the sense that it allows us to send quantum information using classical channels given that some prep work is done first. In other words, if we are struggling to create a true quantum channel, we can scaffold that with a channel that depends on Bell pairs. We can summarize the protocol in the following diagram:



"Send qubit by using classical bits"



8.2 Superdense Coding

Superdense coding is the dual to what quantum teleportation is. This time, the goal is to send classical information by sending a qubit.

Suppose Alice wants to send 2 bits of classical information to Bob. We will see here that there is a way to communicate these bits of classical information by just sending a single qubit. Again, Alice and Bob will prepare by creating an entangled pair of qubits $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Depending on the information Alice wants to send, she performs the following operations:

- "00": $I \to$ $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$ Bell state. $|\Phi^+\rangle$
- "01": $Z \to$ $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$ $|\Phi^-\rangle$
- "10": $X \to$ $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle) = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$ $|\Psi^+\rangle$
- "11": $ZX \to$ $\frac{1}{\sqrt{2}}(|10\rangle - |01\rangle) = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{bmatrix}$ $|\Psi^-\rangle$

Bell Basis

Question 73. Verify that the above vectors form an orthonormal basis.

$\langle \Phi | \Psi \rangle = 0$ b.c. nonzero indices are zero in the other one.

$\langle \Phi^+ | \Phi^- \rangle = \frac{1}{2} - \frac{1}{2} = 0$

$\langle \Psi^+ | \Psi^- \rangle = \frac{1}{2} - \frac{1}{2} = 0$

Since the four states above are orthonormal, we can perform a measurement in that basis! One way to do model this is to just compute the inner product with another state, but in practice we are often constrained by what bases we are able to measure in. Let's see how we might measure in an alternate basis given the constraint that we can only measure in the standard basis.

Question 74. Design a circuit that converts between the standard basis and the basis from the previous problem.

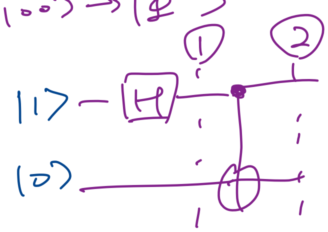
$|\langle \Phi^+|\psi\rangle|^2 = ?$

S.B. → B.B.

00 → Φ^+
01 → Φ^-
10 → Ψ^+
11 → Ψ^-

some reversible mapping (bijection)

$|00\rangle \to |\Phi^+\rangle$



① $\frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)$

② $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |\Phi^+\rangle$

$|00\rangle - |10\rangle$

$|00\rangle - |11\rangle$

$|\Phi^+\rangle = \overbrace{(CNOT)(H\otimes I)}|00\rangle$

Question 75. Use the previous problem to describe how Bob could measure in the Bell basis assuming he only has a measurement device that can only measure in the standard basis.

$|\langle \Phi^+|\psi\rangle|$

$= |\langle 00|\underbrace{(H\otimes I)^\dagger (CNOT)^\dagger|\psi\rangle}_{\varphi}|$

In contrast to teleportation, suppose we find in the future that it becomes so cheap to send a qubit over a quantum channel, such that it costs pretty much the same as sending a classical bit. In this case, the superdense coding protocol can allow us to send double the amount of information with one transmission!